A Posteriori Error Estimation for Discontinuous Galerkin Approximations of Hyperbolic Systems

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Abstract. This article considers *a posteriori* error estimation of specified functionals for first-order systems of conservation laws discretized using the discontinuous Galerkin (DG) finite element method. Using duality techniques, we derive exact error representation formulas for both linear and nonlinear functionals given an associated bilinear or nonlinear variational form. Weighted residual approximations of the exact error representation formula are then proposed and numerically evaluated for Ringleb flow, an exact solution of the 2-D Euler equations.

1 Introduction

A frequent objective in numerically solving partial differential equations is the subsequent calculation of certain derived quantities of particular interest, e.g., aerodynamic lift and drag coefficients, stress intensity factors, etc. Consequently, there is considerable interest in constructing *a posteriori* error estimates for such derived quantities so as to improve the reliability and efficiency of numerical computations. For an introduction to *a posteriori* error analysis see Eriksson et al. [9], related work by Estep et al. [13], Parashivoiu et al. [15], and the recent report of Oden and Prudhomme [14]. For hyperbolic problems and applications in fluid mechanics see Johnson et al. [12], Giles et al. [10], Becker and Rannacher [4] and Süli [16].

This article revisits the topic of *a posteriori* error estimation of prescribed functionals with special emphasis and consideration given to nonlinear systems of conservation laws discretized using the discontinuous Galerkin (DG) finite method, see for example Johnson and Pitkäranta [11], Bey and Oden [5], and Cockburn et al. [7,8]. In a departure from this previous work, our DG formulation for systems of conservation laws uses entropy symmetrization variables as discussed in detail in the companion papers by the second author [3,2,1].

In Section 2, we briefly review the abstract model for *a posteriori* error estimation of functionals. Next, we consider the DG method for nonlinear systems of conservation laws and derive concrete error estimates in terms of element residual and weight formulas. Section 4 numerically assesses the sharpness of these estimates for the specific example of Ringleb flow which has a known exact solution via hodograph transformation.

2 *A Posteriori* Error Estimation of Functionals

Abstract model problem. In this section, we give an abstract presentation of *a posteriori* error estimation for functionals based on duality techniques. Consider the following abstract variational problem: find $\boldsymbol{u} \in X$ such that

$$\mathcal{A}(\boldsymbol{g}; \boldsymbol{u}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in X, \tag{2.1}$$

and the corresponding discrete problem: find $\boldsymbol{u}_h \in X_h$ such that

$$\mathcal{A}(\boldsymbol{g}; \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in X_h. \tag{2.2}$$

Here X is a suitable function space, $X_h \subset X$ is a discrete space, for instance, discontinuous piecewise polynomials of degree k, and $\boldsymbol{g}$ some prescribed data. Note that boundary conditions are ***weakly*** imposed in the variational statement thus permitting both $\boldsymbol{u}$ and $\boldsymbol{v}$ to reside in X. For brevity, we sometimes write $\mathcal{A}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \mathcal{A}(\boldsymbol{g}; \boldsymbol{u}_h, \boldsymbol{v}_h)$. Our objective is to estimate the error

$$M(\boldsymbol{u}) - M(\boldsymbol{u}_h), \tag{2.3}$$

in a given functional $M(\cdot)$. The first step is to derive an error representation formula.

Error representation: linear case. We first assume that $\mathcal{A}(\cdot,\cdot)$ and $M(\cdot)$ are both linear. To derive a representation formula for the error (2.3), we introduce the dual problem: find $\boldsymbol{\Phi} \in X$ such that

$$\mathcal{A}(\boldsymbol{v}, \boldsymbol{\Phi}) = M(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in X. \tag{2.4}$$

Setting $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_h$ in (2.4) yields

$$\begin{aligned}
M(\boldsymbol{u}) - M(\boldsymbol{u}_h) &= M(\boldsymbol{u} - \boldsymbol{u}_h) && \text{(linearity of } M\text{)}\\
&= \mathcal{A}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\Phi}) && \text{(2.4)}\\
&= \mathcal{A}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi}) && \text{(orthogonality)}\\
&= \mathcal{A}(\boldsymbol{u}, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi}) - \mathcal{A}(\boldsymbol{u}_h, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi}) && \text{(linearity of } \mathcal{A}\text{)}\\
&= -\mathcal{A}(\boldsymbol{u}_h, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi}) && \text{(2.1),}
\end{aligned}$$

where $\pi_h \boldsymbol{\Phi} \in X_h \subset X$ is an interpolant of $\boldsymbol{\Phi}$. Thus we have the error representation formula

$$M(\boldsymbol{u}) - M(\boldsymbol{u}_h) = -\mathcal{A}(\boldsymbol{g}; \boldsymbol{u}_h, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi}). \tag{2.5}$$

3 *A Posteriori* Error Estimates for the DG Method

First-order nonlinear system. Consider the prototype conservation law problem: find $\boldsymbol{u} : \Omega \to \mathbb{R}^m$ such that

$$\begin{aligned} L(\boldsymbol{u}) = \boldsymbol{f}^i(\boldsymbol{u})_{,x_i} &= 0 \quad \text{in } \Omega, \\ \tilde{A}^-(\boldsymbol{n};\boldsymbol{g},\boldsymbol{u})(\boldsymbol{g}-\boldsymbol{u}) &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.1}$$

where Ω is a domain in $\mathbb{R}^d$ with boundary Γ with local exterior normal vector $\boldsymbol{n}$ and $\tilde{A}(\boldsymbol{n};\boldsymbol{u}) \equiv \boldsymbol{n}_i \boldsymbol{f}^i_{,\boldsymbol{u}}$ is the flux Jacobian matrix. In addition, $\bar{A}(\boldsymbol{n};\boldsymbol{g},\boldsymbol{u})$ denotes the mean value matrix obtained from the path integration

$$\bar{A}(\boldsymbol{n};\boldsymbol{s},\boldsymbol{t}) = \int_0^1 \tilde{A}\left(\boldsymbol{n};\boldsymbol{t}+\theta(\boldsymbol{s}-\boldsymbol{t})\right)\, d\theta \tag{3.2}$$

and $P^{\pm}(\boldsymbol{n};\boldsymbol{s},\boldsymbol{t})$ the associated characteristic projectors. Throughout, we assume that $\boldsymbol{u}$ denotes the symmetrization variables so that the matrices $\tilde{A}$ are necessarily symmetric.

Next, consider a finite element tessellation $\mathcal{T}$ of Ω composed of nonoverlapping elements T_i, $\mathcal{T} = \cup T_i$, $T_i \cap T_j = \emptyset$, $i \neq j$ and $\Gamma_{\mathcal{T}}$ the tessellated boundary. The prototype system can be restated in variational form[1]: find $\boldsymbol{u} \in X$ such that $\forall\, \boldsymbol{v} \in X$

$$\begin{aligned} \mathcal{A}(\boldsymbol{g};\boldsymbol{u},\boldsymbol{v}) = \sum_{T\in\mathcal{T}} \Big((L(\boldsymbol{u}),\boldsymbol{v})_T &+ \langle \bar{A}^-(\boldsymbol{n};\boldsymbol{g},\boldsymbol{u})(\boldsymbol{g}-\boldsymbol{u}),\boldsymbol{v}\rangle_{\partial T\cap\Gamma_{\mathcal{T}}} \\ &+ \langle P^-(\boldsymbol{n};\boldsymbol{u}_-,\boldsymbol{u}_+)[\boldsymbol{f}(\boldsymbol{n};\boldsymbol{u})]_-^+,\boldsymbol{v}_-\rangle_{\partial T\setminus\Gamma_{\mathcal{T}}} \Big) \end{aligned} \tag{3.3}$$

Note that other mathematically equivalent formulations are possible by grouping together terms element-wise and edge-wise. The above particular grouping has been chosen as it reflects a discrete balance of conserved quantities on an element-by-element basis. In Section 4, we briefly revisit the possibility of alternate groupings although our numerical results show that the element-wise grouping presented above yields superior estimates.

***A posteriori* error estimate residuals and weights.** A straightforward application of Cauchy-Schwarz inequality (with $\tilde{A}_0$ introduced from entropy symmetrization theory for dimensional consistency) (3.3) yields the following

[1] In actual implementations it is desirable to use an integrated-by-parts form (see for example [3,2]) so that exact discrete conservation is achieved on elements with inexact quadrature and/or inexact path integration (3.2).

Error representation: nonlinear case. Consider now the case of a nonlinear variational form $\mathcal{A}(\cdot,\cdot)$ and functional $M(\cdot)$. To perform the analysis given above, we introduce the following mean value linearizations

$$\mathcal{A}(g;\boldsymbol{u},\boldsymbol{v}) = \mathcal{A}(g;\boldsymbol{u}_h,\boldsymbol{v}) + \overline{\mathcal{A}}(g,\boldsymbol{u}_h,\boldsymbol{u};\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in X \tag{2.6}$$
$$M(\boldsymbol{u}) = M(\boldsymbol{u}_h) + \overline{M}(\boldsymbol{u}_h,\boldsymbol{u};\boldsymbol{u}-\boldsymbol{u}_h), \tag{2.7}$$

and the dual linearized problem: find $\boldsymbol{\Phi} \in X$ such that

$$\overline{\mathcal{A}}(g,\boldsymbol{u}_h,\boldsymbol{u};\boldsymbol{v},\boldsymbol{\Phi}) = \overline{M}(\boldsymbol{u}_h,\boldsymbol{u};\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in X. \tag{2.8}$$

In addition, we have the following orthogonality relation

$$\overline{\mathcal{A}}(g,\boldsymbol{u}_h,\boldsymbol{u};\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in X_h. \tag{2.9}$$

Proceeding in the same fashion as above, using simplified notation for brevity,

$$\begin{aligned}
M(\boldsymbol{u}) - M(\boldsymbol{u}_h) &= \overline{M}(\boldsymbol{u}-\boldsymbol{u}_h) && (2.7)\\
&= \overline{\mathcal{A}}(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\Phi}) && (2.8)\\
&= \overline{\mathcal{A}}(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi}) && (2.9)\\
&= \mathcal{A}(\boldsymbol{u},\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi}) - \mathcal{A}(\boldsymbol{u}_h,\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi}) && (2.6)\\
&= -\mathcal{A}(\boldsymbol{u}_h,\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi}), && (2.1)
\end{aligned}$$

thus yielding the following final error representation formula

$$M(\boldsymbol{u}) - M(\boldsymbol{u}_h) = -\mathcal{A}(g;\boldsymbol{u}_h,\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi}). \tag{2.10}$$

Abstract a posteriori error estimates. Starting from (2.5) or (2.10), we derive various error estimates by estimating the right hand side of (2.10) using standard inequalities. Later, the sharpness of these inequalities is numerically assessed. Consider the following sequence of direct estimates

$$\begin{aligned}
|M(\boldsymbol{u}) - M(\boldsymbol{u}_h)| &= \Big|\sum_T \mathcal{A}_T(g;\boldsymbol{u}_h,\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi})\Big| && (2.11)\\
&\le \sum_T |\mathcal{A}_T(g;\boldsymbol{u}_h,\boldsymbol{\Phi}-\pi_h\boldsymbol{\Phi})| && (2.12)\\
&\le \sum_T R_T(\boldsymbol{u}_h)\cdot W_T(\boldsymbol{\Phi}), && (2.13)
\end{aligned}$$

where $\mathcal{A}_T(\cdot,\cdot)$ denotes the restriction of $\mathcal{A}(\cdot,\cdot)$ to the element T. Further $R_T(\boldsymbol{u}_h)$ is a computable estimate of the residual of $\boldsymbol{u}_h$ on T, and $W_T(\boldsymbol{\Phi})$ is a weight on T describing the local influence of $\boldsymbol{\Phi}$, both are defined below.

element residuals R_T and weights W_T for use in (2.13)

$$R_T(u_h) = \begin{bmatrix} \|L(u_h)\|_{\tilde{A}_0^{-1},T} \\ \|P^-(n; u_{-,h}u_{+,h})[f(n;u_h)]_-^+\|_{\tilde{A}_0^{-1},\partial T\backslash\Gamma_T} \\ \|\tilde{A}^-(n,g;u_h)(g-u_h)\|_{\tilde{A}_0^{-1},\Gamma_T} \end{bmatrix} \tag{3.4}$$

$$W_T(\Phi) = \begin{bmatrix} \|\Phi - \pi_h\Phi\|_{\tilde{A}_0,T} \\ \|\Phi - \pi_h\Phi\|_{\tilde{A}_0,\partial T\backslash\Gamma_T} \\ \|\Phi - \pi_h\Phi\|_{\tilde{A}_0,\Gamma_T} \end{bmatrix} \tag{3.5}$$

Approximating the dual problem. The weight formulas (3.5) require the calculation of the quantity $\Phi - \pi_h\Phi$ from the dual problem which requires a priori knowledge of both u and u_h for use in the mean value linearizations (2.6) and (2.7). Since u is not generally known, we supplant this calculation with the approximate discrete counterpart to (2.8): find $\Phi_h \in X_h$ such that

$$\overline{\mathcal{A}}(g, u_h, u_h; v_h, \Phi_h) = \overline{M}(u_h, u_h; v_h) \quad \forall\, v \in X_h. \tag{3.6}$$

Observe that $\overline{\mathcal{A}}(g, u_h, u_h; v, \Phi_h)$ and $\overline{M}(u_h, u_h; v_h)$ are precisely the Jacobian linearized forms of the respective operators. Using the techniques described in Barth [3.2], exact Jacobian derivatives of the DG scheme for systems of conservation laws have been derived and used in all subsequent calculations. We have investigated the computation of the needed dual solution terms using two different techniques:

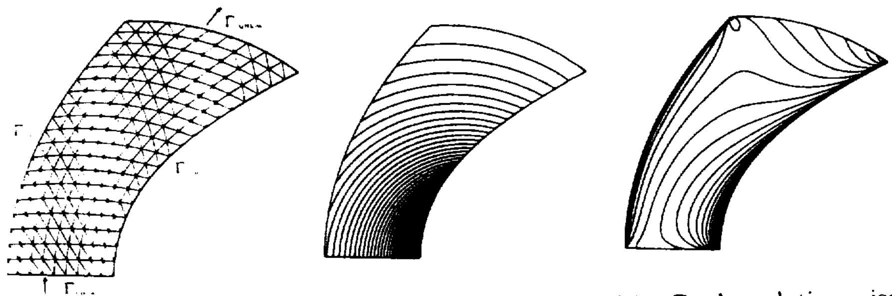
(1) High-order approximation. Suppose $u_h^{(k)}$ denotes a numerical solution computed in $X_h^{(k)}$. Embed $u_h^{(k)}$ in $X_h^{(l)}$, $l > k$ and approximate $\Phi - \pi_h^{(k)}\Phi \approx \Phi_h^{(l)} - \pi_h^{(k)}\Phi_h^{(l)}$. This technique is employed in the calculations given below.

(2) Recovery post-processing. Let $\mathcal{R}_h^{(l)}\Phi_h^{(k)} : X_h^{(k)} \mapsto X_h^{(l)}$ denote a recovery operator, approximate $\Phi - \pi_h^{(k)}\Phi \approx \mathcal{R}_h^{(l)}\Phi_h^{(k)} - \Phi^{(k)}$, $l > k$. Recovery operators based on local compact supported least-squares fitting are considered in a forthcoming report by the present authors.

4 Numerical Results

To evaluate the accuracy of the error representation formulas given in Sect. 2, Ringleb flow (an exact solution of the 2-D Euler equations obtained via hodograph transformation, see [6]) is computed in the channel geometry shown in Fig. 4.1(a). Next, the vertical force component exerted on the channel walls is computed from the functional

$$M_\Psi(u) = \int_{\Gamma_{\mathrm{Wall}}} (\Psi \cdot \mathrm{n})\, p(u)\, dl \tag{4.7}$$



(a) Coarsest triangulation (342 elements). (b) Primal solution iso-density contours. (c) Dual solution iso-density contours.

Fig. 4.1. Ringleb flow test problem. Primal and dual solutions calculated using the DG discretization with cubic elements for the vertical force functional $M_{\Psi}(u)$.

with $p(\boldsymbol{u})$ the fluid pressure and Ψ a constant vertical vector. Iso-density contours of the Ringleb primal and dual solutions are given in Figs. 4.1(b-c).

We now evaluate the validity and sharpness of the error estimate formulas (2.11)-(2.13) and (3.5). In Fig. 4.2 we graph for constant (a) and linear (b) approximation: ($\circ$) the exact error; ($\times$) estimate (2.11); (Δ) estimate (2.12); (∇) estimate obtained from element-edge form of (3.3); ($\Box$) estimate (3.5). In all cases the dual problem is defined by (3.6) and solved using cubic polynomials. The difference between ($\circ$) and ($\times$) is caused by linearization, i.e., replacing $\boldsymbol{u}$ in (2.8) by $\boldsymbol{u}_h$ to get (3.6). This appears to be a very small error. Next, the more significant loss due to use of the Triangle Inequality is graphed in (Δ). This prevents cancellation between elements. Further error is introduced via Cauchy-Schwarz ($\Box$) thus preventing cancellation within the element. Finally, note that the element based estimate (Δ) is notably superior to the element-edge based estimate (∇), where in the latter case contributions are grouped together in such a way that element conservation is violated. Based on our numerical experimentation, we propose the adaptive method:

- Evaluate a *stopping criterion* via $|A(\boldsymbol{u}_h, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi})_\Omega|$.
- Evaluate an *adaptation criterion* via $|A(\boldsymbol{u}_h, \boldsymbol{\Phi} - \pi_h \boldsymbol{\Phi})_T|$.

In addition, the adaptation criterion may be further improved by the use of sharpened variants of the Triangle and Cauchy-Schwarz Inequalities. We consider these topics further in a forthcoming paper.

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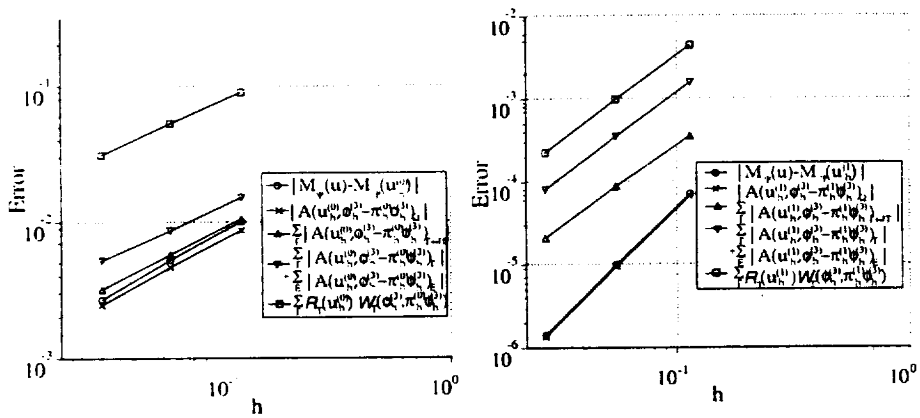

(a) Piecewise constant elements. (b) Piecewise linear elements.

Fig. 4.2. Ringleb flow problem. Sharpness of error estimate inequalities for the vertical force functional (4.7).

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